Optimal Risk Sharing in Insurance Networks

An Application to Asset-Liability Management

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(based on joint work with Anna-Maria Hamm and Stefan Weber)

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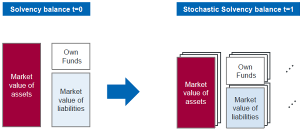
Motivation and Outline

- Classical capital regulation is based on a standalone view of financial firms.
- Companies can structure their business in the form of corporate networks; this may distort capital requirements, if regulatory risk measures are not convex.
- Our case studies will illustrate that for downside risk measures of V@R-type corporate networks can swipe (all) downside risk under the carpet.
- Topics of the talk:
  - I. Review of capital regulation
  - II. Network risk and risk sharing
  - III. Application to Asset-Liability Management

Part I: Capital Regulation

Regulatory and Solvency Capital

- Role of capital:
  - Buffer for potential losses
  - that protects customers, policy holders and other counterparties
- Calculation principles in a nutshell:
  - Market-consistent valuation of all assets and liabilities
  - Stochastic balance sheet projections capturing the random evolution of the firm's equity over a given time horizon



  - Computation of the capital requirement based on the prognosis distribution
- Simple example: Solvency II
  - SCR = Solvency Capital Requirement
  - Key goal: Limit the one-year probability of ruin to at most 0.5%

Monetary Risk Measures and Capital Requirements

- Framework:
  - Model for one time period as in Solvency II: $t = 0, 1$
  - $\mathcal{X}$ is the space of financial positions at time 1
- Monetary risk measure $\rho : \mathcal{X} \to \mathbb{R}$:
  - Inverse monotonicity: If $X \geq Y$, then $\rho(X) \leq \rho(Y)$.
  - Cash invariance: If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.

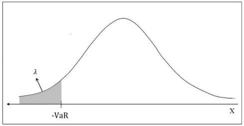
  A monetary risk measure is a statistic that summarizes certain properties of random future balance sheet.
- Capital requirement:
  - A position $X \in \mathcal{A}$ is acceptable, if $\rho(X) \leq 0$.
    The collection $\mathcal{A}$ of all acceptable positions is the acceptance set.
  - ρ is a capital requirement, i. e.

$$\rho(X) = \inf \{ m \in \mathbb{R} : X + m \in \mathcal{A} \} .$$

Value at Risk and Beyond

- Example: Value at Risk at level $\lambda \in (0,1)$

$$\mathrm{V@R}_\lambda(X) = \inf\{m \in \mathbb{R} : P[X+m<0] \leq \lambda\} = -q_X^+(\lambda)$$



where q_X^+ denotes the upper quantile function of X.

  - V@R is not a convex risk measure and may thus penalize diversification. Moreover, $\mathrm{V@R}_\alpha$ neglects extreme losses that occur with small probability.
  - These deficiencies were a major reason to develop a systematic theory of coherent and convex risk measures, cf. Artzner et al. (1999) and Föllmer&Schied (2002).
  - Basis of capital requirements in the regulation scheme Solvency II

- Alternative: Average Value at Risk/Expected Shortfall (coherent risk measure)

$$\mathrm{AV@R}_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda \mathrm{V@R}_\alpha(X)\,d\alpha$$

  - AV@R accounts for extreme losses and provides incentives for diversification.
  - AV@R coincides with the Tail Value at Risk for continuous random variables (see, e. g., Acerbi&Tasche (2000)):

$$\mathrm{TV@R}_\lambda(X) := \mathbb{E}[-X \mid -X > \mathrm{V@R}_\lambda(X)]$$

  - Basis of capital requirements in the Swiss Solvency Test (SST)

SCR and Solvency II

- Recital 64 of Directive 2009/138/EC:

  *The **Solvency Capital Requirement** should be determined as the **economic capital** to be held by insurance and reinsurance undertakings in order to ensure that ruin occurs no more often than once in every 200 cases or, alternatively, that those undertakings will still be in a position, **with a probability of at least 99.5 %, to meet their obligations to policy holders and beneficiaries over the following 12 months**.*

- SCR in a simplified Internal Model:
  - Time: $t = 0, 1$ (no discounting on the one-year horizon)
  - Value of assets: A_t, $t = 0, 1$
  - Value of liabilities: L_t, $t = 0, 1$
  - Equity (NAV): $E_t = A_t - L_t$, $t = 0, 1$

  $$P[E_1 < 0] \leq 0.005 \Leftrightarrow E_1 \in \mathcal{A}_{\mathrm{V@R}_{0.005}} \Leftrightarrow \mathrm{SCR}_{\mathcal{A}}(E_1) := \mathrm{V@R}_{0.005}(E_1 - E_0) \leq E_0.$$

- Canonical SCR definition in the context of Solvency II:
  - $\mathrm{SCR}_{\mathcal{A}}(E_1) = \mathrm{V@R}_{0.005}(E_1 - E_0) = E_0 + \mathrm{V@R}_{0.005}(E_1)$
  - Interpretation: Coverage ratio above 100% if and only if $P[E_1 < 0] \leq 0.005$

Solvency II - Regulatory Requirements

- Directive 2009/138/EC, Article 101(3), or §97(2) VAG:

  With respect to existing business, it shall cover only unexpected losses. It shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99,5% over a one-year period.

- SCR in practice: Mean Value at Risk

$$\mathrm{SCR}_{\mathrm{mean}}(E_1) := \mathrm{V@R}_{0.005}(E_1 - \mathbb{E}[E_1]) = \mathbb{E}[E_1] + \mathrm{V@R}_{0.005}(E_1)$$

- Remarks:
  - Both definitions are consistent to specific regulatory requirements, but lead however to different solvency capital requirements.
  - In a Gaussian setting, risk with respect to $\mathrm{SCR}_{\mathrm{mean}}$ can be aggregated by square-root-formula. This is a key assumption of the SII Standard Formula.

$$\mathrm{SCR}_{\mathrm{mean}}(X+Y) = \sqrt{\mathrm{SCR}_{\mathrm{mean}}(X)^2 + \mathrm{SCR}_{\mathrm{mean}}(Y)^2 + 2\rho \mathrm{SCR}_{\mathrm{mean}}(X)\mathrm{SCR}_{\mathrm{mean}}(Y)}$$

    - $X \sim \mathcal{N}(\mu, \sigma^2) \Rightarrow \mathrm{V@R}_\lambda(X) = -\mu - \Phi^{-1}(\lambda)\sigma$, $\mathrm{SCR}_{\mathrm{mean}}(X) = -\Phi^{-1}(\lambda)\sigma$
    - $\sigma^2(X+Y) = \sigma^2(X) + \sigma^2(Y) + 2\rho\sigma(X)\sigma(Y)$ for all $X, Y \in L^2$ with correlation ρ
  - Both $\mathrm{SCR}_{\mathrm{mean}}$ and $\mathrm{SCR}_{\mathcal{A}}$ inherit the deficiencies of V@R as a measure of risk!

Alternative Risk Measures

- Let ρ denote a monetary (convex) risk measure with acceptance set $\mathcal{A}$:

$$\mathrm{SCR}_{\mathcal{A}}(E_1) := \rho(E_1 - E_0)$$
$$\mathrm{SCR}_{\mathrm{mean}}(E_1) := \rho(E_1 - \mathbb{E}[E_1])$$

- Note that $\rho(E_1) \leq 0 \;\Leftrightarrow\; E_1 \in \mathcal{A} \;\Leftrightarrow\; \mathrm{SCR}_{\mathcal{A}}(E_1) \leq E_0$.
- Suitable examples: Coherent risk measure AV@R (SST & Basel III) and Expectiles.
- Comparable to the specifications of the Target Capital of the Swiss Solvency Test (see Technical document on the SST).

Part II: Network Risk and Risk Sharing

Network Risk

- Insurance firm is not consolidated, but forms a corporate network:
  - Subentities $i = 1, 2, \dots, n$,
  - individually regulated according to risk measures ρ^i
- The total network balance sheet can be split among the sub-entities using legally binding transfer agreements:

$$(E_t^i)_{i=1,2,\dots,n} \text{ with } E_t = \sum_{i=1}^{n} E_t^i, \quad t = 0, 1$$

- Total SCR of the network:

$$\sum_{i=1}^{n} \mathrm{SCR}^i_{\mathcal{A}}(E_1^i) = E_0 + \sum_{i=1}^{n} \rho^i(E_1^i)$$

$$\sum_{i=1}^{n} \mathrm{SCR}^i_{\mathrm{mean}}(E_1^i) = \mathbb{E}[E_1] + \sum_{i=1}^{n} \rho^i(E_1^i)$$

Optimal Risk Sharing

- The corporate network can design optimal transfer agreements in order to minimize the SCR.
- This leads to the following optimal risk sharing problem:

$$\square_{i=1}^{n}\rho^{i}(E_1) := \inf\left\{\sum_{i=1}^{n}\rho^{i}\left(E_1^{i}\right)\,\middle|\,\sum_{i=1}^{n}E_1^{i}=E_1,\ E_1^{1},\dots,E_1^{n}\in\mathcal{X}\right\}$$

  - This is also known as inf-convolution, introduced by Barrieu&El Karoui (2005) and Barrieu&El Karoui (2008).
  - Rich literature on optimal risk sharing, e. g., Galchion (2010), Jouini, Schachermayer&Touzi (2008), Embrechts, Liu&Wang (2018)
- Corresponding solvency capital requirements:

$$\square_{i=1}^{n}\mathrm{SCR}_{\mathcal{A}}^{i}(E_1) := E_0+\square_{i=1}^{n}\rho^{i}(E_1) \quad \text{and} \quad \square_{i=1}^{n}\mathrm{SCR}_{\mathrm{mean}}^{i}(E_1) := \mathbb{E}[E_1]+\square_{i=1}^{n}\rho^{i}(E_1)$$

Example 1: Risk Sharing and Coherent Risk Measures

- As a first example, consider the case that $\rho^i = \rho$, $i = 1, 2, \ldots, n$.
- If ρ is coherent, then

$$\rho(E_1) = \rho\left(\sum_{i=1}^{n} E_1^i\right) \leq \sum_{i=1}^{n} \rho(E_1^i).$$

  - This lower bound is attained for $E_1^i = \alpha^i E_1$, $i = 1, \ldots, n$, with $\alpha^1 + \ldots + \alpha^n = 1$.
  - In particular, it is optimal to allocate the total net asset value to one entity, e. g. the holding company.
- This is the situation that holds for the Swiss Solvency Test which is based on the coherent risk measure AV@R.
- In contrast, V@R – the basis of Solvency II – is not coherent.

Example 2: Risk Sharing and Value at Risk

- Suppose now that $\rho^i = \mathrm{V@R}_{\alpha_i}$, $\alpha_1, \ldots, \alpha_n \in (0,1)$, then

$$\square_{i=1}^{n} \mathrm{V@R}_{\alpha_i}(E_1) = \mathrm{V@R}_{\sum_{i=1}^{n} \alpha_i}(E_1).$$

  see, e. g., Embrechts, Liu&Wang (2018)
- In particular: $\square_{i=1}^{n} \mathrm{V@R}_{\alpha}(E_1) = \mathrm{V@R}_{n \cdot \alpha}(E_1)$.
- The optimal allocation $(E_1^i)_{i=1,2,\ldots,n}$ can explicitly be computed.

For Value at Risk, appropriate network structures and transfer agreements permit to swipe all downside risk under the rug!

- Embrechts, Liu&Wang (2018) show that the same problem occurs for the Range Value at Risk RV@R suggested by Cont, Deguest&Scandolo (2010):
  - $\mathrm{RV@R}_{\alpha,\beta}(X) = \frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} \mathrm{V@R}_{\gamma}(X)\, d\gamma$ for $\alpha, \beta > 0$ with $\alpha + \beta \leq 1$
  - $\square_{i=1}^{n} \mathrm{RV@R}_{\alpha_i,\beta_i}(E_1) = \mathrm{RV@R}_{\sum_{i=1}^{n} \alpha_i, \max\{\beta_1,\ldots,\beta_n\}}(E_1)$.
- More generally, Weber (2018) analyzes the risk sharing problem for all V@R-type risk measures, extending the results of Embrechts, Liu&Wang (2018).

Example 3: Risk Sharing and Distortion Risk Measures (1)

- Class of distortion risk measure including V@R, AV@R, RV@R (Weber (2018)):
  - Specific distortion function: $g : [0,1] \to [0,1]$ increasing with
  $$g(x) = 0 \text{ for } x \in [0,\alpha],\ g(x) > 0 \text{ for } x \in (\alpha,1],\ g(1) = 1$$
  - Distortion risk measures $\rho^g(X) := \int (-X) dc^g$ defined as the Choquet integral with respect to a capacity $c^g(A) := g(P[A])$, $A \in \mathcal{F}$
  - ρ^g is coherent if and only if g is concave.
  - Alternative representation as mixtures: $\rho^g(X) = \int_{[0,1]} \mathrm{V@R}_\lambda(X)\, g(d\lambda)$
- If the parameter $\alpha > 0$, then ρ^g is called a V@R-*type distortion risk measure*.
  - Interpretation: $\rho^g(X) = \int_{[\alpha,1]} \mathrm{V@R}_\lambda(X)\, g(d\lambda)$ does not depend on any properties of the tail of X beyond its V@R at level α.
  - V@R and RV@R are V@R-type distortion risk measures, AV@R is not.

Example 3: Risk Sharing and Distortion Risk Measures (2)

- Key results in Weber (2018):
  - Let $g^1, g^2, \ldots, g^n$ denote left-continuous distortion functions with finitely many jumps and parameters $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0, 1)$ and define $d = \sum_{i=1}^n \alpha_i$.
  - Construction of an optimal risk sharing such that:
    - For $d < 1$, the allocation swipes all losses beyond $\mathrm{V@R}_d(X)$ under the rug.
    - If $d \geq 1$, then $\square_{i=1}^n \rho^i(E_1) = -\operatorname{ess\,sup} E_1$, corresponding to the best case.
  - V@R-type risk measures swipe losses under the carpet.
- Additional contribution in Hamm, Knispel&Weber (2018): Fair allocation from single firms perspective

Part III: Application to Asset-Liability Management

Outline

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- Networks can implement various (static) asset allocation strategies over a one-year time horizon. We analyze three case studies of different complexity:
  1. Assets are modeled by a Black-Scholes market, liabilities are deterministic.
  2. Liabilities may be random; different types of dependence between assets and liabilities are investigated.
  3. An additional left-tailed asset is available.
- For these cases, we quantify the impact of the number n of sub-entities in the network on the network's minimal risk $\square_{i=1}^{n} \rho^{i}\left(E_1\right)$ and on the SCR.
- We demonstrate how ALM can further reduce the minimal network risk.
- We focus on three different risk measures: V@R, AV@R and RV@R.

Technical Remarks

- Parameterization of risk measures:
  - Within the network all firms use the same risk measure:
    (a) $\rho^i = \text{V@R}_\alpha$, $\alpha \in (0,1)$, for all $i = 1, \dots, n$,
    (b) $\rho^i = \text{AV@R}_\beta$, $\beta \in (0,1)$, for all $i = 1, \dots, n$,
    (c) $\rho^i = \text{RV@R}_{\gamma,\epsilon}$, $\gamma, \epsilon \in (0,1)$, for all $i = 1, \dots, n$.
  - For V@R_α, we choose the level $\alpha = 0.1$, and we fix $\gamma = 0.05$ for the RV@R.
  - The remaining parameters β, ϵ are calibrated such that for $X \sim \mathcal{N}(0,1)$

$$\text{V@R}_\alpha(X) = \text{AV@R}_\beta(X) = \text{RV@R}_{\gamma,\epsilon}(X).$$

  - Summary of parameter:

V@R_α	AV@R_β	$\text{RV@R}_{\gamma,\epsilon}$
$\alpha = 0.1$	$\beta = 0.2456$	$\gamma = 0.05$, $\epsilon = 0.1072$

- Monte-Carlo simulation: 500, 000 simulations
  - Asset distributions are modified by setting asset values above the 99.95%-quantile to the 99.95%-quantile.
  - Liability distributions are modified by setting liability values above the 99.95%-quantile to the 99.95%-quantile, and below the 0.05%-quantile to the 0.05%-quantile

General Asset-Liability Model

- Time: ALM model with finite time horizon 1
- Assets:
  - Financial market with a finite number $K \geq 1$ of liquidly traded assets
  - A_t^k, $t \in [0, 1]$, price of one share of asset $k = 1, \dots, K$
- Liabilities: L_t consolidated liabilities at time $t \in [0, 1]$
- Static asset allocation strategy in the period $t \in [0, 1]$:
  - δ^k fraction of the total asset amount of the balance sheet invested in asset k
  - Asset allocation strategy $\delta \in \mathbb{R}^K$ with $\delta^k \geq 0$ and $\sum_{k=1}^K \delta^k = 1$
  - Numbers of shares held in the assets $k = 1, \dots, K$:

$$\eta^k(\delta) = \delta^k \cdot \frac{E_0 + L_0}{A_0^k}$$

- Net Asset Value:

$$E_t(\delta) = \sum_{k=1}^{K} \eta^k(\delta) A_t^k - L_t, \quad t \in [0, 1]$$

Basis model

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- Asset Model: Black-Scholes market on $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}, P)$
  - Savings account: $A_t^1 = \exp(rt)$, $t \in [0,1]$, with interest rate r
  - Stock: $A_t^2 = A_0^2 \exp(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t)$, $t \in [0,1]$, $(W_t)_{t\in[0,1]}$ Wiener process
- Liability model:
  - The insurance network sells a pure endowment with maturity 1 only.
  - The network's premium income in $t = 0$ is denoted by π. The liabilities are deterministic, and the actuarial interest rate is assumed to be zero, i. e.,

$$L_t = \pi, \quad t \in [0,1].$$

- Net Asset Value:

$$E_t(\delta) = \eta^1(\delta)A_t^1 + \eta^2(\delta)A_t^2 - L_t = \eta^1(\delta)A_t^1 + \eta^2(\delta)A_t^2 - \pi \quad (t \in [0,1])$$

- Parameterization:
  - Asset side: $r = 0$, $A_0^2 = 30$, drift $\mu = \ln(35/30) \approx 0.1542$ (i. e., $\mathbb{E}[A_1^2] = 35$), volatility $\sigma = 0.2$, asset value bounded by its 99.95%-quantile 66.2512
  - Liability side: $\pi = 90$, i. e. $L_0 = L_1 = \pi = 90$
  - $E_0(\delta) = 30$, i. e., total asset amount of the balance sheet $E_0(\delta) + L_0 = 120$
  - Asset allocation: $\delta^1 = 0.75$, $\delta^2 = 0.25$, i. e., $\eta^1(\delta) = 90$, $\eta^2(\delta) = 1$, $E_1(\delta) = A_1^2$

Unsophisticated Network vs. Sophisticated Network

- Numerical results: Basis ALM model with deterministic liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \text{V@R}_\alpha^i \, (E_1(\delta))$	$\square_{i=1}^n \text{SCR}_{\mathcal{A}}^i \, (E_1(\delta))$	$\square_{i=1}^n \text{SCR}_{\text{mean}}^i \, (E_1(\delta))$
$n = 1$	**34.9982**	**-26.5577**	**3.4423**	**8.4405**
$n = 5$	**34.9982**	**-34.3060**	**-4.3060**	**0.6922**
$n = 10$	**34.9982**	**-66.2512**	**-36.2512**	**-31.2530**

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \text{AV@R}_\beta^i \, (E_1(\delta))$	$\square_{i=1}^n \text{SCR}_{\mathcal{A}}^i \, (E_1(\delta))$	$\square_{i=1}^n \text{SCR}_{\text{mean}}^i \, (E_1(\delta))$
$n = 1, 5, 10$	**34.9982**	**-26.6784**	**3.3216**	**8.3198**

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \text{RV@R}_{\gamma,\epsilon}^i \, (E_1(\delta))$	$\square_{i=1}^n \text{SCR}_{\mathcal{A}}^i \, (E_1(\delta))$	$\square_{i=1}^n \text{SCR}_{\text{mean}}^i \, (E_1(\delta))$
$n = 1$	**34.9982**	**-26.5722**	**3.4278**	**8.4260**
$n = 5$	**34.9982**	**-30.9523**	**-0.9523**	**4.0459**
$n = 10$	**34.9982**	**-35.2473**	**-5.2473**	**-0.2491**

- Conclusion:
  - For V@R and RV@R, downside risk can be reduced significantly by optimal capital transfers that hide the tail risk.
    - For n sufficiently large, $\square_{i=1}^n \rho^i(E_1(\delta)) = -\operatorname{ess\,sup} E_1(\delta) = -66.2512$.
    - This requires $n \cdot \alpha \geq 1$ for V@R$_\alpha$. For V@R$_{0.1}$, this condition is already satisfied for $n \geq 10$, and the simulations provide the expected result.
  - In contrast, for the coherent risk measure AV@R, optimal risk sharing does, of course, not reduce the risk capital.

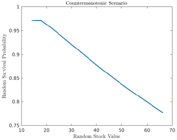
Random Liabilities

- We extend the basis ALM model by including random liabilities.
- Notation:
  - $L > 0$ sum insured
  - p_x^* one-year actuarial survival probability for insured persons aged x
  - p_x one-year random survival probability for insured persons aged x
- Assumptions:
  - p_x^* is the best estimate of the random survival probability, i. e. $\mathbb{E}[p_x] = p_x^*$.
  - p_x^* does not yet include any margin for unexpected losses.
- Liabilities: $\pi = L \cdot p_x^*$, $L_1 = L \cdot p_x = \frac{p_x}{p_x^*}\pi$
- Network's random equity at time $t = 1$:

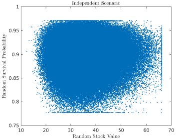
$$E_1(\delta) = \eta^1(\delta) + \eta^2(\delta)A_1^2 - L_1 = \eta^1(\delta) + \eta^2(\delta)A_1^2 - \frac{p_x}{p_x^*}\pi$$

- Parameterization:
  - $L = 100$, $p_x^* = 0.9$, $p_x \sim \text{Beta}(90, 10)$, i. e., $\mathbb{E}[p_x] = p_x^* = 0.9$, $\mathbb{E}[L_1] = \pi = L_0$.
  - Asset allocation: $\delta^1 = 0.8382$, $\delta^2 = 0.1618$ (calibrated such that for a network with a single firm only and for independent assets and liabilities $\text{V@R}_\alpha(E_1(\delta))$ coincides with the basis ALM model)

Random Liabilities - Dependence



(a)



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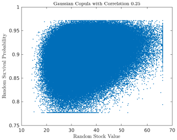
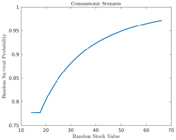


(c)



(d)

(a) Countermonotonicity

(b) Independence

(c) Gaussian Copula with correlation 0.25

(d) Comonotonicity

Motivation:

- Illustrate the implications of particularly extreme forms of dependence
- Gaussian copula according to the specifications of the Solvency II Standard Formula
- Countermonotonic assets and liabilities are problematic, since high insurance claims occur together with low asset values and yield a low book value of equity of insurers.

Random Liabilities and Optimal Risk Sharing

(a) Countermonotonic stock and liabilities:

	$\square_{i=1}^{n} \mathrm{V@R}_{\alpha}^{i}\ (E_1(\delta))$	$\square_{i=1}^{n} \mathrm{AV@R}_{\beta}^{i}\ (E_1(\delta))$	$\square_{i=1}^{n} \mathrm{RV@R}_{\gamma,\epsilon}^{i}\ (E_1(\delta))$
$n = 1$	**-24.1537**	**-24.3001**	**-24.1789**
$n = 5$	**-32.5189**	**-24.3001**	**-28.8983**
$n = 10$	**-65.7126**	**-24.3001**	**-33.5348**

(b) Independent stock and liabilities:

	$\square_{i=1}^{n} \mathrm{V@R}_{\alpha}^{i}\ (E_1(\delta))$	$\square_{i=1}^{n} \mathrm{AV@R}_{\beta}^{i}\ (E_1(\delta))$	$\square_{i=1}^{n} \mathrm{RV@R}_{\gamma,\epsilon}^{i}\ (E_1(\delta))$
$n = 1$	**-26.5578**	**-26.6353**	**-26.5684**
$n = 5$	**-32.8451**	**-26.6353**	**-30.1805**
$n = 10$	**-65.7126**	**-26.6353**	**-33.5769**

(c) Gaussian Copula with correlation 0.25:

	$\square_{i=1}^{n} \mathrm{V@R}_{\alpha}^{i}\ (E_1(\delta))$	$\square_{i=1}^{n} \mathrm{AV@R}_{\beta}^{i}\ (E_1(\delta))$	$\square_{i=1}^{n} \mathrm{RV@R}_{\gamma,\epsilon}^{i}\ (E_1(\delta))$
$n = 1$	**-27.3255**	**-27.3935**	**-27.3377**
$n = 5$	**-32.9015**	**-27.3935**	**-30.5547**
$n = 10$	**-64.0618**	**-27.3935**	**-33.5492**

(d) Comonotonic stock and liabilities:

	$\square_{i=1}^{n} \mathrm{V@R}_{\alpha}^{i}\ (E_1(\delta))$	$\square_{i=1}^{n} \mathrm{AV@R}_{\beta}^{i}\ (E_1(\delta))$	$\square_{i=1}^{n} \mathrm{RV@R}_{\gamma,\epsilon}^{i}\ (E_1(\delta))$
$n = 1$	**-31.7546**	**-31.7879**	**-31.7601**
$n = 5$	**-32.5588**	**-31.7879**	**-32.0290**
$n = 10$	**-46.2791**	**-31.7879**	**-32.7668**

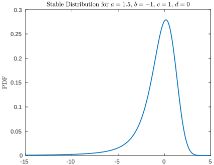
- Conclusion:
  - For a single firm only and for all three risk measures V@R, AV@R and RV@R, the risk capital $\rho(E_1(\delta))$ reflects the riskyness of the different dependence structures.
  - For V@R-type risk measures, optimal sharing has stronger effects for more dangerous dependency structures.

Left-Tailed Asset

- We extend the basis ALM model by including a third left-tailed asset.

$$A_t^3 = A_0^3 \exp(\zeta t) + Z - \mathbb{E}[Z], \quad t \in (0,1],$$

where the initial value $A_0^3 > 0$ is a fixed constant, $\zeta > 0$ is a rate of exponential growth, and Z is a random variable with stable distribution $\mathcal{S}(a,b,c,d)$.



- This asset is characterized by a skewed distribution with the possibility of losses and – in comparison to the stock – a higher downside risk.
- Parameterization:
  - $A_0^3 = 1$, $\zeta = 0.3$, $Z \sim \mathcal{S}(1.5, -1, 1, 0)$ independent from $(A_t^2)_{t\in[0,1]}$
  - Note that $\mathbb{E}[A_1^3/A_0^3] \approx \exp(\zeta) > \exp(\mu) \approx \mathbb{E}[A_1^2/A_0^2]$ for the parameters $\zeta = 0.3$ and $\mu = 0.1542$, i. e., the expected return of the left-tailed asset exceeds the expected return of the stock, compensating for the higher risk of this position.

Fixed Asset Allocation Including a Left-Tailed Asset

- Asset allocation: $\delta^1 = 0.73901, \delta^2 = 0.2510, \delta^3 = 0.01$ (calibrated such that for a single firm, $\text{V@R}_\alpha(E_1(\delta))$ coincides with the basis case)
- Numerical results: ALM model with left-tailed asset and deterministic liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \text{V@R}^i_\alpha\ (E_1(\delta))$	$\square_{i=1}^n \text{AV@R}^i_\beta\ (E_1(\delta))$	$\square_{i=1}^n \text{RV@R}^i_{\gamma,\epsilon}\ (E_1(\delta))$
$n = 1$	**35.4378**	**-26.5577**	**-25.4473**	**-26.5512**
$n = 5$	**35.4378**	**-35.1833**	**-25.4473**	**-31.5879**
$n = 10$	**35.4378**	**-71.8246**	**-25.4473**	**-36.1717**

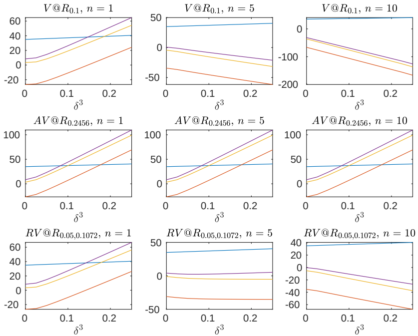
- Numerical results: Basis ALM model with deterministic liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \text{V@R}^i_\alpha\ (E_1(\delta))$	$\square_{i=1}^n \text{AV@R}^i_\beta\ (E_1(\delta))$	$\square_{i=1}^n \text{RV@R}^i_{\gamma,\epsilon}\ (E_1(\delta))$
$n = 1$	**34.9982**	**-26.5577**	**-26.6784**	**-26.5722**
$n = 5$	**34.9982**	**-34.3060**	**-26.6784**	**-30.9523**
$n = 10$	**34.9982**	**-66.2512**	**-26.6784**	**-35.2473**

- Conclusion:
  - Optimal capital transfers within a sophisticated network hide the downside risk, if capital regulation is based on V@R and RV@R.
  - The decay of risk is stronger in comparison to the basis ALM model.

Optimizing the Asset Allocation

- Let us fix the fraction $\delta^1 = 0.75$ invested in the savings account and vary the fraction δ^3 held in the left-tailed asset in the range $[0, 0.25]$.



- $\mathbb{E}[E_1(\delta)]$ in blue
- $\square_{i=1}^n \rho^i\,(E_1(\delta))$ in red
- $\square_{i=1}^n \mathrm{SCR}^i_{\mathcal{A}}\,(E_1(\delta))$ in yellow
- $\square_{i=1}^n \mathrm{SCR}^i_{\mathrm{mean}}\,(E_1(\delta))$ in purple

- Conclusion:
  - For $n = 1$ all risk measures indicate that investments into the left-tailed asset increase risk, in line with the true risk profile.
  - If n is large, assets with a fat left tail are particularly attractive for V@R-type risk measures, since downside risk can be hidden particularly easily.

Conclusion

- Network risk management with V@R-type risk measures permits to hide tail risk by using appropriate transfer agreements between the entities of a corporate network.
- If the number of subentities n is sufficiently large, the network can design a capital allocation such that the optimal network risk $\square_{i=1}^{n}\rho^{i}(E_1)$ coincides with $-\operatorname{ess\,sup} E_1$, corresponding to the best case scenario.
- Case studies show that V@R-type risk measures provide incentives for risky ALM management, i. e. – from a regulatory point of view – for risk mismanagement.
- In contrast, if risk management is based on the coherent risk measure average value at risk, downside risk cannot be hidden and misleading incentives are not present.

Selected References

Asimit, A. V., A. M. Badescu & Tsanakas, A. (2013): Optimal Risk Transfers in Insurance Groups, *European Actuarial Journal* 3(1), 159-190.

Embrechts, P., H. Liu & Wang, R. (2018): Quantile-Based Risk Sharing, *Operations Research* 66(4), 936-949.

Hamm, A.-M., Knispel, T. & Weber, S, (2019): Optimal Risk Sharing in Insurance Networks An Application to Asset-Liability Management.

Weber, S. (2018): Solvency II, or How to Sweep the Downside Risk Under the Carpet, *Insurance: Mathematics and Economics* 82, 191-200.

Thank you for your attention!

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Appendix: V@R-Type Risk Measures

Distortion functions for the risk measures V@R, AV@R and RV@R for $\alpha, \beta > 0$ with $\alpha + \beta \leq 1$

Risk Measure	V@R_α	AV@R_β	$\text{RV@R}_{\alpha,\beta}$
g(x) =	$\begin{cases} 0, & 0 \leq x \leq \alpha \\ 1, & \alpha < x \end{cases}$	$\begin{cases} \frac{x}{\beta}, & 0 \leq x \leq \beta \\ 1, & \beta < x \end{cases}$	$\begin{cases} 0, & 0 \leq x \leq \alpha \\ \frac{x-\alpha}{\beta}, & \alpha < x \leq \alpha+\beta \\ 1, & \alpha+\beta < x \end{cases}$
Type	V@R-**type**	**Not** V@R-**type**	V@R-**type**

Appendix: Optimal Risk Sharing (1)

Theorem (Weber (2018), Theorem 2.4)

Let $E_1 \in L^\infty$ *and* $n \in \mathbb{N}$. *By* $g^1, g^2, \ldots, g^n$ *we denote left-continuous distortion functions with parameters* $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0,1)$ *and define* $d = \sum_{i=1}^n \alpha_i$. *We set* $\rho^i = \rho^{g_i}$, *i. e.,* ρ^i *is the distortion risk measure associated with the distortion function* g^i, $i = 1, 2, \ldots, n$. *Define the left-continuous functions*

$$f = \min\left\{\widehat{g^1}, \widehat{g^2}, \ldots, \widehat{g^n}\right\}, \qquad g(x) = \begin{cases} 0, & 0 \le x \le d \wedge 1, \\ f(x-d), & d \wedge 1 < x \le 1 \end{cases}$$

Note that $g \equiv 0$, *if* $d \ge 1$. *In particular,* g *is not necessarily a distortion function with* $g(1) = 1$. *We set* $\mathrm{V@R}_\lambda := \mathrm{V@R}_1 = -\operatorname{ess\,sup}$ *for* $\lambda \ge 1$.

1. *There exist* $E_1^1, E_1^2, \ldots, E_1^n \in L^\infty$ *such that* $\sum_{i=1}^n E_1^i = E_1$ *and*

$$\sum_{i=1}^n \rho^i(E_1^i) = \int_{[0,1]} \mathrm{V@R}_\lambda(E_1) g(d\lambda) + (g(1) - 1) \operatorname{ess\,sup} E_1.$$

   If $d \ge 1$, *this equation can be simplified and we obtain*

$$\sum_{i=1}^n \rho^i(E_1^i) = -\operatorname{ess\,sup} E_1.$$

Appendix: Optimal Risk Sharing (2)

Theorem (Weber (2018), Theorem 2.4 (continued))

2. *The allocation* $(E_1^i)_{i=1,2,\dots,n}$ *can be constructed as follows. Let* $Y := E_1 - \operatorname{ess\,sup} E_1 \leq 0$*. There exists a random variable* U*, uniformly distributed on* $[0,1]$*, such that* $Y = -\mathrm{V@R}_U(Y)$*. For* $i = 1, 2, \dots, n$*, we set*

$$r_i(\lambda) = \begin{cases} 1, & i = \inf\{j : \hat{g}_j(1-\lambda) = f(1-\lambda)\}, \\ 0, & \textit{else}, \end{cases}$$

$(\lambda \in [0,1])$ *and* $R_i(y) = -\int_0^{|y|} r_i(\lambda)d\lambda$*. We define* $\tilde{Y} = Y \cdot \mathbf{1}_{\{U \geq d\}}$ *and* $\tilde{E}_1^i = R_i(\tilde{Y})$*. For* $i = 1, 2, \dots, n$*, we set*

$$E_1^i = Y \cdot \mathbf{1}_{\left\{\sum_{l=1}^{i-1} \alpha_l \,\leq\, U \,<\, \sum_{l=1}^{i} \alpha_l\right\}} + \tilde{E}_1^i + \frac{\operatorname{ess\,sup} E_1}{n}$$

If $d \geq 1$*, this equation can be simplified and we obtain*

$$E_1^i = Y \cdot \mathbf{1}_{\left\{\sum_{l=1}^{i-1} \alpha_l \,\leq\, U \,<\, \sum_{l=1}^{i} \alpha_l\right\}} + \frac{\operatorname{ess\,sup} E_1}{n}$$

Appendix: Stable Distribution

Definition

A random variable Z has a *stable distribution* $\mathcal{S}(a, b, c, d)$ with parameters $a \in (0, 2]$, $b \in [-1, 1]$, $c \in (0, \infty)$, $d \in \mathbb{R}$, i. e., $Z \sim \mathcal{S}(a, b, c, d)$, if its characteristic function is given by

$$\mathbb{E}\left[e^{isZ}\right] = \begin{cases} \exp\left(-c^{\alpha}|s|^{a}\left[1 + ib\,\mathrm{sign}(s)\tan\frac{\pi a}{2}\left(\left(c|s|^{1-a} - 1\right)\right)\right] + ids\right), & a \neq 1, \\ \exp\left(-c|s|\left[1 + ib\,\mathrm{sign}(s)\tan\frac{2}{\pi}\left(c|s|\right)\right] + ids\right), & a = 1. \end{cases}$$